From Four Colour Conjecture to Four Colour Theorem – an analysis by means of Zentralblatt MATH

In their everyday routines, mathematicians every now and then will have to deal with results which in some cases resulted from the engagement of whole generations of researchers who helped to solve a tough problem. Not always in their lifetime the problems attracted the same amount of attention. Can the MATH database help to illustrate these aspects? We take the Four Colour Theorem as an example for which we show how the interest in a topic has evolved over a long period.

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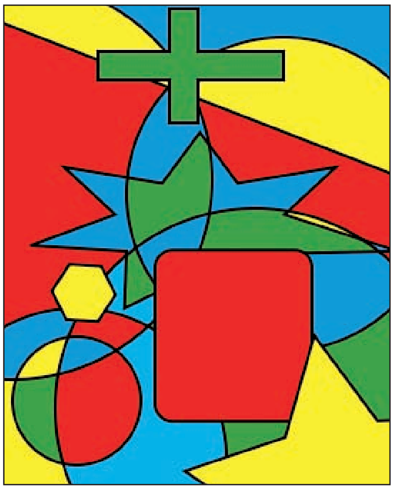
The development of mathematics can be traced, among other things, by means of the many problems which served as a starting point for research. Fruitful problems do not only attract interest in a narrow field; trying to find a solution to them often gives an insight into other areas of mathematics. Thus, interconnections arise which often are not anticipated in that form. This effect arises whenever the straight path to a solution is too pebbly and equivalent formulations of the problem are found which at best make it possible to achieve the goal in a simpler way in a different area of mathematics.

In the mathematical community huge problems are sometimes a dominating topic for many years, even decades. During their lifetime they attract different degrees of attention, sometimes more and sometimes less. The longer the problems are around, the stronger and the wider is their influence on their mathematical environment. The fact that in some cases they generate activities in several rooms of the "mathematics building", sometimes even on different floors, made the total of these problems the driving force behind mathematical research. And the more widespread these the problems are prior to their solution, the more the various areas will profit from the solution once it has been found.

In his book "The mathematical century: The 30 greatest problems of the last 100 years" [Princeton University Press (2004; Zentralblatt MATH 1065.00003)], P. Odifreddi presents the greatest problems of the last 100 years and describes their impact, both on the area in which they were generated and on mathematics as a whole.
Let us quote a few of the problems mentioned there: Einstein's general relativity theory, Gorenstein's classification of simple groups, Bieberbach's symmetry groups, Hale's solution of Kepler's problem, Wiley's proof of Fermat's last theorem, the four colour theorem of Appel and Haken, Kolmogorov's axiomatization, etc.
As mentioned before, these problems were not paid the same amount of interest during all of their lives. We therefore consider it of interest whether the development of the attention paid to a problem can be documented in a rather simple way by means of the number of papers on it reviewed in Zentralblatt MATH.

Let us choose the Four Colour Problem as our example. It was first formulated when in the year 1852 Francis Guthrie, who just had graduated from University College in London, in a letter to his younger brother Frederick, who was still a student, asked him to determine whether it was mathematically proven that every map painted on a sheet of paper can be coloured with only four colours in such a way that nowhere two countries of the same colour share a borderline. The Four Colour Conjecture was formulated, however, it would take 124 years until a final proof could be furnished. Their teacher, Prof. De Morgan, could not provide any proof and thus it remained for a long time. In 1878, A. Cayley brought the problem forward to the London Mathematical Society, but he was not able to solve it. One year later, A.B. Kempe presented a proof which in his opinion was true. Kempe's idea was to find a set of configurations that was unavoidable in a normal map. At the same time he was looking for reducible configurations, where a configuration is said to be reducible if it cannot be part of a minimal five-coloured map. In the year 1890 Kempe's attempt of a proof was recognized to be faulty by P.J. Heawood who also noted that it would be very difficult to "repair" the proof. He tried to transfer the problem of colouring a map on a plane surface to that of colouring a map on more complex surfaces, which seemed to be easier to solve. Heawood worked on the problem for 60 years. In the course of his efforts regarding the four colour problem he proved the five colour theorem which says that each normal map can be coloured with five colours. You can find a nice proof of this theorem in the book "Proofs from THE BOOK" of M. Aigner and G.M. Ziegler. Shortly after 1910, new ways were pursued, and new reducible configurations were found. D. Birkhoff improved Kempe's approach by showing that even larger configurations than those described by Kempe are reducible. Now first concrete results were presented. In 1922, P. Franklin found out that maps with 25 countries maximum can be coloured with four colours. In 1950, it was already known that maps may even contain 36 countries to remain four colourable.

In 1937, H. Whitney formulated a numerical equivalent to the four colour problem, another attempt to attack it by making a detour. There was an arithmetic approach, even a four colour formula was given (J. Nuut, 1931). And the probability of the correctness of the four colour theorem was investigated (E. Krahn, 1932). After all these attempts in the 1920s and 1930s, interest in the four colour problem dwindled a little. In fact the main reason for this was World War II, but also after the war the problem only gradually regained its place in the focus of whole workgroups. New approaches were designed, the failures of earlier ones were investigated and recognized. Many of the well known graph theorists took care of the problem. In 1969, H. Heesch, who had been working on the problem since 1936, wrote a whole book on the four colour problem. He suggested that he would be able to prove the theorem by finding an unavoidable set of reducible configurations. His work paved the way for the generation of a computer program which could perform the operations that were necessary to determine



this unavoidable set, which would have been impossible by hand. K. Appel and W. Haken finally were the ones who were successful with a computer program of that kind in 1977 . They found 1936 unavoidable configurations. Later they were able to reduce this set to 1476 members, and in 1996, N. Robertson, D. Sanders, P. Seymour, and R. Thomas could bring down the number of unavoidable reducible configurations to 633.

With Appel's and Haken's proof in 1977, one of the big problems was for the first time solved by means of computer programs. This approach was rather unusual at that time, a classical proof with pencil and paper is also in this case still longed for. However, checking a great number of cases in a reasonable time seems to be only possible by computer support.

You can find all articles and books mentioned above when you perform a search in the database ZMATH which comprises data from both Zentralblatt MATH and Jahrbuch über die Fortschritte der Mathematik and thus reaches back to the year 1868. With many little queries you can collect valuable information on those books and articles which in their times paved the way for the solution of the four colour problem. You may also find those papers whose approach did not lead into the right direction. After a few queries one notices that times of high interest in the four colour problem were followed by times of lower interest and vice versa. It is interesting that a very simple query gives us a first idea of the intensity of interest paid to the problem in individual periods. As the term "four colour problem" usually appears in the title of the relevant papers, or otherwise certainly in the review, it seems sensible to carry out the following little query in "Basic Index" (BI):

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[ bi: „four colo*“ | „vierfarb*“ |
„quatre couleur“| „quattro colori*“]
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The result over the complete database comprises 488 papers. If you distribute these into time bins, for example decades, you get the following table:

Time period	Hits
<=1900	6
1901-1910	3
1911-1920	3
1921-1930	23
1931-1940	46
1941-1950	11
1951-1960	7
1961-1970	22
1971-1980	83
1981-1990	82
1991-2000	129
2001-today	83

The result of this little query already allows you to notice that after the problem had been formulated in 1852 quite some time passed by until it attracted reasonable attention. Some important contributions from the second decade of the last century have lead to a growing interest in the four colour problem, which even increased in the 1920s and 1930s. But the complexity of this problem, the formulation of which is so simple, seems to have reduced the amount of interest paid to it thereafter. Only when it became more and more common to prove mathematical problems with the help of computers, the four colour problem regained its place in the focus of researchers. By the numbers of papers reviewed in ZMATH in the 1970s one can clearly notice the increase of interest; the proof from 1977 falls in this decade. The following years saw, besides attempts to improve or simplify the proof, many papers which dealt with the effects on those areas of mathematics for which equivalent formulations had emerged. Certainly the number of papers in the database concerning the four colour problem/theorem is much higher than the 488 shown as a result of our simple query, but the general conclusion made on the basis of this result should not change.
This is a nice little example which demonstrates how one can easily get an overview over development paths in mathematics.

Selected literature:

P.J. Heawood, On the four colour map theorem, Quart. J. 29, 270-285 (1898; JFM 28.0426.01),

D. Birkhoff, The reduciblity of maps, Am. J. Math. 35, 115-128 (1913; JFM 44.0568.01),

J. Nuut, Über die Anzahl der Lösungen der Vierfarbenaufgabe, Acta. Univ. Dorpat 15, No. 3, 52 p. (1929; JFM 57.1516.03),

H. Whitney, The coloring of graphs, Ann. of Math. (2) 33, 688-718 (1932; JFM 58.0606.01),

P.J. Heawood, On extended congruences connected with the four colour math theorem, Proceedings L. M. S. (2) 33, 253-286 (1932; JFM 58.1203.03),

P. Heegard, Über die Heawoodschen Kongruenzen, Norsk Mat. Forenings Skrifter 2, No. 6, 47-54 (1933; JFM 59.0555.06),

K. Wagner, Bemerkungen zum Vierfarbenproblem, Jber. DMV 46, 26-32 (1936, JFM 62.0656.02),

W.T. Tutte, On the four colour conjecture, Proc. Lond. Math. Soc., II. Ser. 50, 137-149 (1948; Zbl 30,08102),

G.A. Dirac, Theorems related to the four colour conjecture. J. Lond. Math. Soc. 29, 143.149 (1954; Zbl 55.17003),

R. Halin, Über eine Satz von K. Wagner zum Vierfarbenproblem, Math. Ann. 153, 47-62 (1963; Zbl 192.30001),

Ø. Ore, The four color problem, Pure and Applied Mathematics, New York-London: Academic Press. 259 p. (1967, Zbl 149.21101),

H. Heesch, Untersuchungen zum Vierfarbenproblem, Mannheim-Wien-Zürich: BI, Hochschultaschenbücher-Verlag, 290 p. (1969; Zbl 187.20904),

H. Whitney and W.T. Tutte, Kempe chains and the four colour problem, Stud. Graph Theory, Part II, MAA Stud. Math. 12, 378-413 (1975; Zbl 328.05107),

K. Appel und W. Haken, Every planar map is four colorable. I: Discharging, Ill. J. Math. 21, 429-490 (1977; Zbl 387.05009), II.: Reduciblity, Ill. J. Math. 21, 491-567 (1977; Zbl 387.05010),

W. Haken, An attempt to understand the four color problem, J. Graph Theory 1, 193-206 (1977; Zbl 387.05011),

J. Mayer, Une propriété des graphes minimaux dans le problème des quatre couleurs, Problèmes combinatoires et théorie des graphes, Orsay 1976, Colloq. Int. CNRS No. 260, 291-295 (1978; Zbl 419.05022),

K. Appel und W. Haken, Every map is four colorable, Contemp. Math, 98. Providence, RI: American Mathematical Society (AMS), 741 p. (1989; Zbl 681.05027),

P.C. Kainen, Is the four color theorem true?, Geombinatorics 3, No. 2, 41-56 (1993; Zbl 854.05045),

R. Fritsch, Der Vierfarbensatz. Geschichte, topologische Grundlagen und Beweisidee. Mannheim: BI Wissenschaftsverlag, 251 p. (1994; Zbl 830.05001),

N. Robertson, D.P. Sanders, P. Seymour, R. Thomas, A new proof of the four colour theorem, Electron. Res. Announc. Am. Math. Soc. 2, No. 1, 17-25 (1996; Zbl 865.05039),

N. Robertson, D.P. Sanders, P. Seymour, R. Thomas, The four color theorem, J. Comb. Theory, Ser. B 70, No. 1, 2-44 (1997; Zbl 883.05056),

M. Aigner und G.M. Ziegler, Proofs from THE BOOK, Berlin, etc.: Springer Verlag (1999; Zbl 905.00001),

E. Gethner und William M. Springer II, How false is Kempe's proof of the four color theorem? Congr. Numeratium 164, 159-175 (2003; Zbl 1050.05049),

G. Gonthier, A computer-checked proof of the four colour theorem, Microsoft Research Cambridge, unveröffentlicht (http://research.microsoft.com/~gonthier/4colproof.pdf)]

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