Where there is music in Zentralblatt – searching for more remote applications of mathematics

Even for topics whose connections with mathematics are not so obvious you will find numerous references in ZMATH. Music is a good example.

Klaus-D. Kiermeier

Since the time of Ancient Greece, mathematicians and non-mathematicians have tried to find connections between music and mathematics. Especially well-known are the findings of Pythagoras of Samos (ca. 580-500 BC) and his followers on the relations of natural numbers, the lengths of a vibrating string, and the pitches produced by this string. The Pythagoreans were interested in the mysticism of numbers and studied these relations by experimenting with a monochord. They realised that a string whose length is subdivided in a ratio represented by a fraction of two natural numbers produces a note that is in “harmony“ with the note produced by the full string: if the ratio is 1:2 then the result is an octave, with 2:3 one gets a perfect fifth, with 3:4 a perfect fourth, etc.

Of particular importance was the discovery of the so-called Pythagorean comma. In all pitch systems that are based on perfect octaves and perfect fifths there is a discrepancy between the interval of seven octaves and the interval of twelve fifths, although both have to be considered as equal in musical terms. This discrepancy results from the difference between $(1/2)^7$ and $(2/3)^{12}$, whose ratio is 524288:531441, and the relevant calculations can already be found in Euclid's work. Musically, this difference makes up approximately an eighth tone.

In musical practice the Pythagorean comma causes serious problems. So in the past numerous approaches were developed to find tunings for instruments that reduce these problems to a minimum. The tuning that today is known best and used most often in European music is the equal temperament or well temperament tuning. This tuning became popular during the baroque era and most notably by “Das wohltemperierte Clavier“ (“The Well-Tempered Clavier“), Bach's grand collection of preludes and fugues that impressively demonstrated the possibility of letting all keys sound equally well. Of course one could also say “equally bad“, since in the equal temperament none of the intervals but the octaves are perfect any more, i.e., the ratios mentioned above are no longer valid.

In the equal temperament every octave is subdivided into twelve half-steps all of which have the same frequency ratio of $2^{\frac{1}{12}}$, where in the terminology above the 2 is to be read as 2:1, i.e., the frequency ratio of an octave. All frequencies of the pitches of the equal tempered twelve-tone scale can be expressed by the geometric sequence

$$f(i) = f_0 \cdot 2^{\frac{i}{12}},$$

where f_0 is a fixed frequency, e.g., the standard pitch a' (440 Hz), and i is the half-step distance of the target note from the note with the frequency f_0. Then, $f(i)$ is the frequency of the target note. The sequence is a geometric one since by its very construction the ratio of two adjacent sequence terms is always the same.

In modern times, Leonhard Euler (1707-1783) was one of the first who tried to use mathematical methods in order to deal with the consonance/dissonance problem. In his work, too, ratios of natural numbers, reflecting frequency ratios of intervals, play an important role. In his paper “Tentamen novae theoriae musicae” of 1739 [in: Opera omnia. Series tertia: Opera physica. Vol. I: Commentationes physicae ad physicam generalem et ad theoriam soni pertinentes. Ediderunt E. Bernoulli, R. Bernoulli, F. Rudio, A. Speiser. Leipzig, B. G. Teubner (1926; JFM 52.0021.07)] Euler defines the following Gradus-suavitatis function Γ [cited after Mazzola, 1990]: Let a be a positive integer. Since every such number can be uniquely factorised into primes, a has a unique representation in the following form:

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_n^{e_n}$$

where $p_1 < p_2 < p_3 < \cdots < p_n$ are a growing sequence of primes and $e_1, e_2, e_3, \ldots, e_n$ are positive integers. Then Euler defines:

$$\Gamma(a) = 1 + \sum_{1 \le k \le n} e_k \, (p_k - 1)$$

and, more general,

$$\Gamma(x/y) = \Gamma(x \cdot y)\,,$$

if (x/y) is a positive reduced fraction.

Inserting fractions that represent ratios of musical intervals into this formula, we obtain the following values (selection):

octave: $\Gamma(1/2) = 2$

fifth: $\Gamma(2/3) = 4$

fourth: $\Gamma(3/4) = 5$

major third: $\Gamma(4/5) = 7$

minor third: $\Gamma(5/6) = 8$

major second: $\Gamma(9/10) = 10$

minor second: $\Gamma(15/16) = 11$

tritone: $\Gamma(32/45) = 14$

According to Euler, these numbers are a measure for the pleasantness of an interval: the smaller the value the more pleasing the interval. Indeed, this is more or less in accordance with our European listening habit, with one exception: the perfect fourth is heard as a dissonance in some contrapuntal and functional harmonic contexts.

Since that time there has been a lot of activity in the area between mathematics and music. This can easily be seen by a simple search in the database Zentralblatt MATH:

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[ bi:music* | bi:musik* | bi:musiq*]
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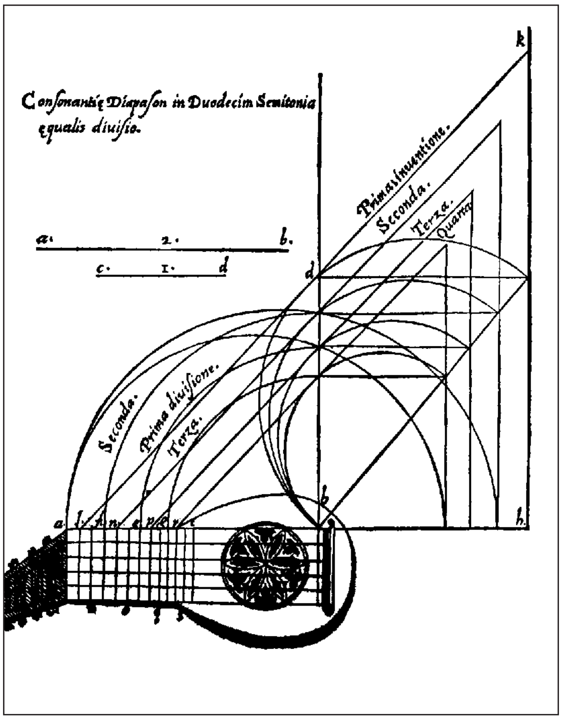

Abb. 26: Geometrical representation of equal tuning from Sopplimenti musicali (1588) by Gioseffo Zarlino

This is to be interpreted as follows:
"bi" means "basic index". In this index of the database all words and word sequences are indexed that appear in any of the fields, be it in the title, in the source, in the review or abstract, or even in the authors field.
The asterisque * means, as usual, truncation, i.e., all words are searched that start with the sequence of characters left from the *. Here, the truncation accounts for the fact that in Zentralblatt MATH there are entries in English, German and French (and rarely also in Italian).
The vertical bar represents a logical "or".

Today (29th February 2008), this query yielded 1199 hits, the most recent ones from 2008, the oldest from 1870. The result shows that even in areas with loose connections to mathematics there is a wealth of literature to be discovered, and its amount has been rapidly growing during the last decades.

A closer look at the result of the search reveals that there is a great variety of topics that are dealt with. Here is a selection of keywords describing some of these topics:

- acoustics (waves, spectra)
- automatic recognition of music pieces, musical styles, musical instruments, performers etc.
- automatic music transcription
- musical scales
- musical tunings
- music perception
- composition of music
- history of music

A great variety of mathematical methods is used here, e.g., number theory, combinatorics, groups, categories, geometry, manifolds, algorithms, neural networks, statistics, fractals, wavelets, differential equations, and much more.

If you are interested in a more specific topic or in particular methods, you can easily refine the above query. For example, the query

```
[ (bi:music* | bi:musik* | bi:musiq*) &
(bi:acoust* | bi:akust* | bi:wave* |
bi:welle* | bi:onde*)]
```

gives 160 hits from the area of musical acoustics. Here, & is a symbol for the logical "and". In a similar way it is possible to restrict a query to a particular time interval:

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[ (bi:"music theory" | bi:musiktheorie) &
(py:1920-1929)]
```

yields, for example, literature on music theory from the 1920s, where "py" means "publication year". If you want to search for a sequence of words in exactly a certain form, then you have to put it into quotation marks.

In any case, if you search in the Zentralblatt MATH database you need to consider that the data were collected over a period of nearly 150 years. So their format is not in all respects homogeneous. This concerns the language (until the middle of the 20th century mostly German, now mostly English) as well as the completeness of the data (with or without reviews or summaries, classification, etc.). Bearing this in mind, you will find a great variety of publications dealing with music and mathematics, historical and brand new ones, mathematically ambitious ones and those for a wider audience. Particularly for a number of recent monographs and textbooks introducing the reader into mathematical music theory, Zentralblatt MATH contains some insightful reviews; see, e.g., Zbl 1051.00007, Zbl 1104.00003, Zbl 1119.00008. You will come directly to the respective database entry if you search for its "accession number":

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[ an: 1104.00003]
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This query gives, e.g.,

Mazzola, Guerino: The topos of music. Geometric logic of concepts, theory, and performance. Basel: Birkhäuser (2002),

one of the most innovative recent works of mathematical music theory.

Sources employed:

Gottwald, Siegfried; Ilgauds, Hans-Joachim; Schlote, Karl-Heinz (eds.): Lexikon bedeutender Mathematiker. Leipzig: Bibliographisches Institut (1990). Zbl 0706.01001

Gurlitt, W.; Eggebrecht, H. H. (eds.): Riemann Musik Lexikon. 12. Aufl. Sachteil. Mainz: B. Schott's Söhne (1967).

Mazzola, Guerino: Geometrie der Töne. Elemente der mathematischen Musiktheorie. Basel: Birkhäuser (1990). Zbl 0729.00008

Wille, Rudolf: "Mathematische Sprache in der Musiktheorie", in: Jahrbuch Überblicke Mathematik 1980, 167-184 (1980). Zbl 0493.00017

Wikipedia entries:
- Pythagorean comma
- Equal temperament

Klaus-D. Kiermeier
FIZ Karlsruhe, Mathematics and Computer Science
Zentralblatt MATH