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A porism for cyclic quadrilaterals, butterfly theorems, and hyperbolic geometry.

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The author focuses on the following theorem obtained by J. Kocik: Theorem 1 . Let p, q, r, s be 4-points lying in a line ℓ. Let C be a circle and choose a point $x \in C$ not collinear with the 4-points. If a chain of four chords starting at x that draw consecutively through p, q, r and s closes (that is, goes back to x, thus the chain forms a cyclic quadrilateral), then a chain for any other choice of starting points passing through p, q, r, s also closes. As a consequence, the configuration space of 4-verices on a round circle making cyclic quadrilaterals has a connected component. In this article, the author gives two proofs of the above theorem: one is based on the cross-ratios on a circle $S^1 \subset \mathbb{R}^2$ (Theorem 2) and the other is due to the property of Möbius transformations of a projective limit circle $tial\mathbb{H}^2$ (Theorem 3). First, the cross-ratio is usually defined by $\mathrm{cr}(a,b;c,d) = \dfrac{a-c}{b-c}/\dfrac{a-d}{b-d}$ for any 4-distinct points. It is invariant under the projective transformation group $\mathrm{PSL}(2,\mathbb{R})$. To prove Theorem 1. using the cross-ratio, the author shows the projective butterfly theorem: Let $p,q,r,s \in \ell$ be intersection points of a line ℓ with a cyclic quadrilateral. If ℓ intersects C in 2-points a,b, then

$$\mathrm{cr}(a,b;p,q) = \mathrm{cr}(a,b;s,r). \tag{1}$$

If ℓ is tangent to C at a point a, then

$$\frac{1}{a-p} - \frac{1}{a-q} = \frac{1}{a-s} - \frac{1}{a-r}. \tag{2}$$

If ℓ and C are disjoint, then

$$\angle paq = \angle sar. \tag{3}$$

Here, a is the tangential point prescribed precisely in the figure (of the paper). Conversely, if C and four points $p,q,r,s \in \ell$ satisfy (1), (2), (3), then for every $x \in C$ there is a closed chain of four chords starting at x and drawing consecutively through p, q, r and s constitutes a cyclic quadrilateral. Secondly, let a circle C and a line ℓ be as above. Choose a point $p \notin C$. If $x \in C$, then define $I_p(x)$ to be the point of C at which the chord of x to p intersects C. Then, the map $I_p : C \to C$ is observed to be a Möbius transformation for each point p of the plane $\mathbb{R}^2$. Here the Möbius transformation group is a subgroup of the projective transformation group $\mathrm{PGL}(3,\mathbb{R})$ preserving the circle S^1 in $\mathbb{RP}^2$, which thus is isomorphic to the hyperbolic group $\mathrm{PO}(2,1)$. The author shows that Theorem 1 is equivalent with Theorem 3. Let $p,q,r,s \in \ell \setminus C$ be any 4-points. If $x \in \ell \setminus C$ satisfies $I_s \circ I_r \circ I_q \circ I_p(x) = x$, then $I_s \circ I_r \circ I_q \circ I_p$ is the identity element in $\mathrm{PO}(2,1)$. The author continues to study a generalization of the above theorem, an n-gone in C (called Castillon's problem).

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