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Sums of two squares. (Summer av to kvadrat.)
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Summary: The characterization of numbers representable as the sum of two squares in terms of their divisors has been known since the 17th century (Girard, Fermat) but not proved until 18th century (Euler). A refinement involving the number $R(n)$ of representations of n as sum of two squares was proved by Jacobi using theta functions. This article starts out by reproducing a very elementary proof due to Heath-Brown of Fermat's characterization of primes of type $4n+1$. Then it turns to discussing problems concerning asymptotic distribution, occurrence in prescribed intervals, infinite occurrences of patterns such as $n, n+h_1$, $n+h_2 \dots h_k$ for fixed h_i (It is shown that $n, n+1$, $n+2$ occurs infinitely often, but of course $n, n+1, n+2, n+3$ may never all be sums of two squares.) Those are compared with the corresponding statements for primes. One may note that the asymptotic behaviour of $B(x) = \sum_{n=a^2+b^2<x} 1$ is more difficult to prove (Landau) than the prime number theorem, but the corresponding problem of $R(x) = \sum_{n=a^2+b^2<x} R(n)$ taking into account the number $R(n)$ of representations of n is much easier to handle and can naturally be linked with elementary asymptotic formulas for the number of lattice points inside circles as observed by Gauss. Finally one may expect that every interval $[x, x+h]$ contains a sum of two squares if $h > C \log x$ for some suitable constant C, but so far it has only been shown for $h > Cx^{\frac{1}{4}}$. But if we relax the condition to almost all such intervals, there is a complete solution.

Classification: F65
Keywords: proof of Fermat's two squares theorem due to Heath-Brown; asymptotic distribution; occurrence in prescribed intervals; analytic number theory