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Petersen, T. Kyle

Eulerian numbers.

Birkhäuser Advanced Texts. Basler Lehrbücher. New York, NY: Birkhäuser/Springer (ISBN 978-1-4939-3090-6/hbk; 978-1-4939-3091-3/ebook). xviii, 456 p. (2015).

This book serves dual purposes. On the one hand, it is a monograph on Eulerian numbers and their generalizations. It goes in depth, four of its chapters marked "supplemental" are clearly for the experts. On the other hand, the book gives an introduction to contemporary enumerative, algebraic and geometric combinatorics, and it can be used as a text at beginning graduate or advanced undergraduate level. The textbook function is supported by a problem set at the end of all non-supplemental chapters. Hints and detailed solutions for the problems make almost a quarter of the volume. Notes on history also help the student. A descent in a permutation is defined as two consecutive entries, where the second is less than the first. Eulerian numbers $\langle {n \atop k} \rangle$ count the number of permutations of n elements with k descents. This was not, however, the way Euler introduced these numbers. Following the work of Bernoulli providing an identity for $\sum_{i=1}^m i^n$ in terms of Bernoulli numbers, Euler found an identity for the polynomial $\sum_{i=1}^m i^n t^i$ in terms of the Eulerian polynomials $\sum_{k=0}^{n-1} \langle {n \atop k} \rangle t^k$, with a particular interest into $t = -1$. It was Riordan who discovered the combinatorial interpretation of Eulerian numbers with descents, and Carlitz made Eulerian numbers well-known through a sequence of papers. The number theoretic aspects of Carlitz' work is not in the focus of this book. The enumeration of 231-avoiding permutations of n elements with k descents yields the Narayana numbers $\frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}$, which sum up to the Catalan number C_n. The analogy of Eulerian and Narayana numbers is followed throughout the book. Another recurring theme is real-rootedness and its relaxation, gamma-nonnegativity. Combinatorial sequences, whose generating polynomials have only realroots, are unimodal. A satisfactory relaxation for palindromic sequences is gamma-nonnegativity, i.e. the sequence is a linear combination of every second row of the Pascal triangle with nonnegative coefficients. The gamma-nonnegativity of Eulerian numbers is due to *D. Foata* and *M. P. Schützenberger* [Théorie géométrique des polynômes eulériens. Berlin-Heidelberg-New York: Springer-Verlag (1970; Zbl 0214.26202)], who were the first to devote a book to Eulerian numbers. (Although the title of this book involve "géométrique", it really means combinatorial.) Geometry gets involved as Eulerian numbers count faces of the permutahedron, while Narayana numbers count faces of the associahedron. Part II of the book is devoted to h-vectors and f-vectors of simplicial complexes. Part III of the book generalizes the concepts of inversion and descent from the symmetric group to finite Coxeter groups and investigates analogues of the Eulerian and Narayana numbers. It is remarkable how far the connections of the topics discussed in the book go: sorting algorithms, compression technique in finite set theory, the number of carries for adding random numbers. This is a well-written text for a good course. *László A. Székely (Columbia)*

Classification: K25

Keywords: Eulerian numbers; Eulerian polynomials; Narayana numbers; Catalan numbers; Euler-Mahonian distributions; pattern-avoiding permutations; descent; major index; noncrossing partitions; gamma-nonnegativity; weak order; permutahedron; hyperplane arrangement; Tamari lattice; associahedron; lattice paths; simplicial complex; barycentric subdivision; h-vectors; f-vectors; Coxeter groups; affine Weyl groups; Steinberg torus

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