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Abstract algebra: structures and applications.

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The book under review provides a comprehensive introduction to the basic concepts, principles, methods, results, and applications of modern abstract algebra. As such it covers the standard material that is usually taught in advanced algebra courses worldwide, however with particular emphasis on the more general structural aspects underlying the whole subject. This means, the book is not a treatise on general algebraic systems and structures, but instead a presentation of the core topics (groups, rings, and fields) following a consistent order that leads the reader to other algebraic structures along the way. In this sense, the term "algebraic structure" is used informally, rather as a methodological organizing principle than a precisely defined theoretical framework. In fact, it signifies that the special order of presentation of the three standard structures follows one unifying scheme: definition of structure, motivation, instructive examples, general properties, subobjects, morphisms, important classes and subclasses of the respective structures, quotient objects, actions and action structures, related algebraic objects, and concrete applications. Indeed, a second guiding principle of this text is to illustrate how abstract algebra can be applied to various other branches of mathematics, which is done by numerous examples, exercises, so-called "investigative projects", and a few extra sections throughout the book. The latter include brief introductions to other, related branches of algebra, thereby inviting the reader to further study in these directions. As for the precise contents, the book consists of thirteen chapters, each of which is organize in several sections. Chapter 1 is of preparatory character and introduces the necessary basic set theory, including sets and mappings, products, operations, relations and equivalence relations, partial orders, and Hasse diagrams. Chapter 2 presents the relevant basics from elementary number theory such as the divisibility properties of integers, the modular arithmetic of $\mathbb{Z}/n\mathbb{Z}$, and the principle of mathematical induction. Chapter 3 is devoted to the elementary properties of groups, with focus on subgroups, group homomorphisms, group presentations, symmetric groups, and applications to cryptography and geometry. Also, a first introduction to semigroups and monoids is given as a structural generalization. Chapter 4 treats quotient groups, including the isomorphism theorems and the fundamental theorem of finitely generated abelian groups, while Chapter 5 discusses the fundamentals of ring theory. Here rings generated by elements, matrix rings, ring homomorphisms, ideals, quotient rings as well as maximal ideals and prime ideals are the main objects of study. Chapter 6 turns to the topic of divisibility in commutative rings, thereby covering rings of fractions, Euclidean domains, unique factorization domains, factorization properties of polynomials, rings of algebraic integers, and the idea of RSA cryptography as a concrete application. Chapter 7 deals with fields and algebraic field extensions, together with explanations of cyclotomic extensions, constructible numbers, splitting fields, the algebraic closure of a field, and the basic properties of finite fields. Chapter 8 returns to group theory by considering group actions, Sylow's theorems, and the idea of group representations, whereas Chapter 9 discusses the classification of groups of given (finite) order via composition series and solvable groups, finite simple groups, semidirect products of groups, and nilpotent groups. Chapter 10 is titled "Modules and algebras" and covers such basic topics like Boolean algebras, free modules and module decomposition, the structure theorem for finitely generated modules over principal ideal domains, the Jordan normal form of linear transformations and its applications, and a brief introduction to path algebras of directed graphs. Elementary Galois theory is the theme of Chapter 11, in which the following standard topics are touched upon: automorphisms of field extensions, the fundamental theorem of Galois theory and its applications, Galois groups of cyclotomic field extensions, discriminants, computing Galois groups of polynomials, field of characteristic $p > 0$, and the problem of solving algebraic equations by radicals. Chapter 12 describes some more specific topics in computational commutative algebra, including introductions to Noetherian rings, multivariable polynomial rings and their role in affine geometry, Hilbert's Nullstellensatz, monomial orders and polynomial division, Gröbner bases, and Buchberger's algorithm with applications, respectively. This chapter ends with a brief introduction to algebraic geometry via affine algebraic sets, the spectrum of a ring, and their Zariski topologies. The final Chapter 13 gives a brief introduction to categories and functors. This chapter is meant as a precise formalization and generalization of the concept of "algebraic structure", which was emphasized (and loosely used) in the previous parts of the book as a unifying methodological principle. The book ends with two appendices on the algebra of complex numbers (A.1) and a list of groups of order at most 24 up to isomorphism (A.2), respectively. Also, there is a very carefully compiled list of notations, a rich bibliography for references and further reading, and a just as helpful index of notions. As already mentioned above, the utmost lucid, detailed and versatile main text comes with a wealth of illustrating examples and very instructive exercises in each single section of the book, and each chapter ends with a section containing project ideas (and hints) to challenge the student to write her or his own investigative or expository papers on related topics. All in all, the book under review is an excellent introduction to the principles of abstract algebra for upper undergraduate and graduate students, and a valuable source for instructors likewise. No doubt, this text is a highly welcome addition to the already existing plethora of primers on abstract algebra in the mathematical literature. *Werner Kleinert (Berlin)*

Classification: H45 H75
Keywords: textbook (algebra); groups; rings; fields; modules; algebras; Galois theory; computational algebra; categories; functors