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On a geometric property of positive definite matrices cone. (English)

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Let $A, B \in \mathbb{C}^n$ be positive definite. For $t \in [0,1]$, the generalized geometric mean $A\#_t B$, of A and B is defined as $A\#_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$. In particular, $A\#_{\frac{1}{2}}B$ henceforth denoted by $A\#B$ is called the geometric mean of A and B. Apparently, this notion first appeared in *W. Pusz* and *S. L. Woronowicz* [Rep. Math. Phys. 8, 159–170 (1975; Zbl 0327.46032)]. Later, *T. Ando* [Linear Algebra Appl. 26, 203–241 (1979; Zbl 0495.15018)] developed many of the fundamental properties of the geometric mean in a systematic manner. *G. E. Trapp* [Linear Multilinear Algebra 16, 113–123 (1984; Zbl 0548.15013)] presented a more recent survey of matrix means. In a recent paper, *T. Ando, C.-K. Li* and *R. Mathias* [Linear Algebra Appl. 385, 305–334 (2004; Zbl 1063.47013)] have shown for positive definite matrices A, B, C, D that $A\#B = C\#D$ implies $(A\#C)\#(B\#D) = A\#B$.

The authors of the paper under review obtain the following generalization (Theorem 3.1) of the above result: In what follows, $X \leq Y$ means that $Y - X$ is positive semi-definite and $0 < X$ means that X is positive definite. Suppose that A, B, C, D are positive definite invertible matrices such that $0 < mI \leq A, B, C, D \leq MI$ for some positive numbers m and M. Suppose also that $A\#_\alpha B = C\#_\alpha D = G$ for some $\alpha \in (0,1)$. Set $\alpha_0 = \min\{\alpha, 1-\alpha\}$ and $h = \frac{M}{m}$. Then for each $\beta \in [0,1]$, the following two matrix inequalities hold:

$$\left\{\frac{(h^{2-\frac{1}{\alpha_0}}+1)^2}{4h^{2-\frac{1}{\alpha_0}}}\right\}^{-\alpha_0} G \leq (A\#_\beta C)\#_\alpha(B\#_\beta D) \leq \left\{\frac{(h^{2-\frac{1}{\alpha_0}}+1)^2}{4h^{2-\frac{1}{\alpha_0}}}\right\}^{-\alpha_0} G.$$

The authors also obtain certain examples and counterexamples related to the main result.

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Classification :

∗15A45 Miscellaneous inequalities involving matrices
15B48
26E60 Means