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Twists of Hessian elliptic curves and cubic fields. (English)

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The paper concerns the family of elliptic curves considered by Hesse in the 1840's:

$$H_\mu : U^3 + V^3 + W^3 = 3\mu UVW$$

where $\mu \neq 1$. The author's main result is the construction of special twists of H_μ, which are defined as follows: Let Ξ be the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -t & -t & 0 \end{pmatrix},$$

and let M be the matrix $xI_3 + y\,\Xi + z\,\Xi^2$ where x, y and z are real-valued indeterminates. Let $\tilde{H}(\mu, t)$ denote the projective curve given by

$$\mathrm{Tr}(M^3) = 3\mu \det(M).$$

The author proves that $\tilde{H}(\mu, t)$ is a twist of H_μ as curves over $\mathbb{Q}$ if $\mu \neq 1$ and $t \neq 0, -27/4$ are rational numbers, and that the two curves become isomorphic over the splitting field of $f_t(X) := X^3 + tX + t$, which is the characteristic polynomial of Ξ. The way the curve $\tilde{H}(\mu, t)$ is constructed immediately allows the following description of the set of rational points on $\tilde{H}(\mu, t)$ if $f_t(X)$ is irreducible over $\mathbb{Q}$: Let f_t be irreducible over $\mathbb{Q}$, let K_t be the field $\mathbb{Q}[X]/(f_t(X))$, and let S be the group

$$\{\eta \in K_t^* : \mathrm{Tr}_{K_t/\mathbb{Q}}(\eta^3) = 3\mu\,\mathrm{N}_{K_t/\mathbb{Q}}(\eta)\}$$

on which $\mathbb{Q}^*$ naturally acts via multiplication. The author establishes a canonical one-to-one correspondence between the set of orbits in S under this action and the set of rational points on $\tilde{H}(\mu, t)$.

Another family $\{H_{\mu,t}\}$ of twists of H_μ is introduced in the paper, which is defined by

$$H_{\mu,t} : 2u^3 + 6d_t uv^2 + w^3 = 3\mu(u^2 - d_t v^3)w$$

where $d_t = -(4t + 27)$; the field $\mathbb{Q}(\sqrt{d_t})$ is an intermediate field of the splitting field of $f_t(X)$. It is trivial that for $\mu \neq 1$ and $t \neq 0, -27/4$, the curve $H_{\mu,t}$ is an elliptic curve and it is a quadratic twist of H_μ as there is a map: $H_{\mu,t} \to H_\mu$ given by

$$[u : v : w] \mapsto [u + \sqrt{d_t}\,v : u - \sqrt{d_t}\,v : w].$$

Hence, $\tilde{H}(\mu, t)$ is isomorphic to $H_{\mu,t}$ over the splitting field of $f_t(X)$. In Theorem 5.1, the author uses an explicit map: $\tilde{H}(\mu, t) \to H_{\mu,t}$ to prove that the isomorphism is in fact over K_t; if f_t is redicible over $\mathbb{Q}$, then K_t is defined to be $\mathbb{Q}$.

For the case of $\mu = 0$, the author provides the following necessary and sufficient condition for $\tilde{H}(\mu, t)$ to have a rational point: $t = -r^3/(r + 1)$ with $r \in \mathbb{Q}$ not equal to 0 or -1, or $\tilde{H}(0,t) \cong \tilde{H}(0,t')$ where $t' = h^3/(2h-1)^2$ for some $h \in \mathbb{Q}$ not equal to 0 or 1/2. In fact, to prove the converse part, the author shows that if $t' = h^3/(2h-1)^2$ for some $h \in \mathbb{Q}$ not equal to 0 or 1/2, then the curve $\tilde{H}(0,t')$ has a rational point, and it is not discussed in the paper whether this can be used to determine whether $\tilde{H}(0,t)$ has a rational point for some examples of t.

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