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On the rational points of the curve $f(X,Y)^q = h(X)g(X,Y)$. (English)

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http://journals.cms.math.ca/cgi-bin/vault/view/poulakis8932
http://journals.cms.math.ca/CMB/
http://www.smc.math.ca/cmb/

Let $\phi : D \to C$ be an unramified morphism of projective smooth curves defined over $\mathbb{Q}$. By the Chevalley-Weil theorem, there is a number field K such that $\phi^{-1}(C(\mathbb{Q})) \subset D(K)$. In the paper under review, the author considers a morphism ψ between two special families of affine curves V and W, and explicitly computes a number field K such that

$$\psi^{-1}(W(\mathbb{Q})) \subset V(K),$$

and uses this setup to completely determine the rational solutions of three examples of curves and two examples of families of curves. This explicit computation is generalized in the recent works *K. Draziotis* and *D. Poulakis* [“Explicit Chevalley-Weil theorem for affine plane curves,” Rocky Mt. J. Math. 39, No. 1, 49–70 (2009; Zbl pre055413170) and An effective version of Chevalley-Weil theorem for projective plane curves, arxiv:0904.3845v1].

Let $F(X,Y) := f(X,Y)^q - h(X)\,g(X,Y)$ be a polynomial in $\mathbb{Z}[X,Y]$ where f, h, and g are fairly general polynomials. Let $h(X) = h_1(X)\,h_2(X)$ be a certain factorization in $\mathbb{Z}[X]$, which always is possible, such that $q \mid \deg(h_1)$. See the paper for details. Consider the varieties

$$V : F(X,Y) = 0,\ T^q = h_1(X),$$
$$W : F(X,Y) = 0.$$

For $q = 2$ or 3, he explicitly computes a finite set S such that the number field K described above is given by $\mathbb{Q}(\sqrt[q]{b} \in S)$. More specifically, he shows that for each $(x,y) \in W(\mathbb{Q})$, there is a $b \in S$ such that the twist $bT^q = h_1(X)$ has a solution (x,t) where $t \in \mathbb{Q}$. Thus, it reduces the problem to that of solving finitely many twists of a superelliptic curve. It seems to the reviewer that the restriction on q in the theorem is only for practical computability purpose and that the theorem can be somewhat explicitly stated for q being a prime number.

He gives a geometric interpretation of this reduction of the original problem in which it is established that the morphism between the projective desingularizations D and C of V and W, respectively, is unramified, and hence, being put into the context of the Chevalley-Weil Theorem. He uses the crucial condition $q \mid \deg(h_1)$ to pull the unramified property out of the towers of the function fields $\overline{\mathbb{Q}}(\mathbb{A})$, $\overline{\mathbb{Q}}(W)$, and $\overline{\mathbb{Q}}(Z)$ where Z is the affine variety $T^q = h_1(X)$, and he uses the condition in the proof of the main theorem for the affine varieties as well.

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14G25 Global ground fields