
Zbl 1042.46001**Alfsen, Erik M.; Shultz, Frederic W.****Geometry of state spaces of operator algebras.** (English)

Boston, MA: Birkhäuser. xiii, 467 p. EUR 98.00/net; sFr. 146.00 (2003). ISBN 0-8176-4319-2/hbk

The important problem of characterizing the state spaces of operator algebras, both associative (C^* and von Neumann) and of Jordan type (JB and JBW-algebras), was raised in the 1950s. During the last three decades, the authors have made significant contributions to this area of research in functional analysis. This monograph features geometric characterizations of state spaces. Notable results obtained in the papers by *E. M. Alfsen* and *F. W. Shultz* [Acta Math. 140, 155–190 (1978; Zbl 0397.46065)], *E. M. Alfsen*, *H. Hanche-Olsen* and *F. W. Shultz* [Acta Math. 144, 267–305 (1980; Zbl 0458.46047)], *B. Iochum* and *F. W. Shultz* [J. Funct. Anal. 50, 317–328 (1983; Zbl 0507.46055)], *E. M. Alfsen* and *F. Shultz* [Commun. Math. Phys. 194, 87–108 (1998; Zbl 0918.46060)] are presented in this book in a unified way, with complete and enlightening proofs and comments. The authors have done a fine work for the mathematical community, providing a valuable toolkit for researchers interested in non-associative structures, self-adjoint operator algebras or areas of functional analysis or mathematical physics where aspects related to convexity and ordered spaces appear.

The prerequisites include selected topics on convexity and ordered structures, C^* and von Neumann algebras and geometric properties of their state spaces (such as orientations), which are the subject of the earlier book of the authors [“State spaces of operator algebras” (Birkhäuser, Basel) (2001; Zbl 0983.46047)]. The main definitions and results are efficiently surveyed in the appendix.

The main body of the book consists of three parts and an appendix. The beginning of Part I gives an account on the structure and representations of Jordan type operator algebras and naturally overlaps with the short monograph of *H. Hanche-Olsen* and *E. Størmer* [“Jordan operator algebras” (Monographs and Studies in Mathematics 21, Pitman) (1984; Zbl 0561.46031)]. The remaining part is devoted to the study of geometric properties of state spaces of such algebras, and of dynamical correspondences and their relation to Connes’s notion of orientation.

Part II develops the study of projective faces, lattices of compressions and a spectral duality (in separating order and norm) between order unit and base norm spaces. This duality plays a key role in constructing operator algebras whose state spaces are affinely homeomorphic to some given compact convex sets. A convex set K is called spectral if it is affinely isomorphic to the distinguished base of a complete base norm space V in spectral duality with the order unit space $A = V^*$.

In Part III, state spaces are characterized. A compact convex set K is said to have the Hilbert ball property if the face generated by each pair of extreme points is a norm exposed Hilbert ball, and the 3-ball property when this face is either a point, a line segment, or a 3-ball. Denote by $A(K)$ the space of continuous affine functions on K . One of the main results states that K is affinely homeomorphic to the state space of a JB-algebra (with the w^* -topology) if and only if the following conditions hold:

- (i) Every $a \in A(K)$ admits a decomposition $a = b - c$ with $b, c \in A(K)^+$ and $b \perp c$;
- (ii) every norm exposed face of K is projective;
- (iii) the σ -convex hull of the extreme points of K is a split face of K and K has the Hilbert ball property.

To define a C^* -structure, additional constraints on K of a geometric nature are required. The main result in this direction is that K is affinely homeomorphic to the state space of a C^* -algebra (with the w^* -topology) if and only if conditions (i) and (ii) together with the following two conditions hold:

- (iii') The σ -convex hull of the extreme points of K is a split face of K , and K has the 3-ball property;
- (iv) K is orientable.

Moreover, if A is a JB-algebra whose state space has the 3-ball property and is orientable, then there is a one-to-one correspondence between global orientations of the state space and C^* -products on $A \oplus iA$.

State spaces of JBW-algebras are characterized, up to affine isomorphism, as compact convex spectral sets which satisfy a geometric property called ellipticity. An additional generalization of the 3-ball property is used to characterize the state spaces of von Neumann algebras.

Florin P. Boca (Urbana-Champaign)

Keywords : Jordan type operator algebras; C^* -algebras; von Neumann algebras; state spaces; compact convex sets; Hilbert ball properties; Connes orientations; dynamical correspondences; projective faces; spectral duality; pure state properties

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- 46L05 General theory of C^* -algebras
- 46L10 General theory of von Neumann algebras
- 46L70 Nonassociative selfadjoint operator algebras
- 46L30 States of C^* -algebras
- 17C65 Jordan structures on Banach spaces and algebras