

io-port 01420782**Aczel, Peter****On relating type theories and set theories.**

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From the introduction: The original motivation for the work described in this paper was to determine the proof-theoretic strength of the type theories implemented in the proof development systems Lego and Coq. These type theories combine the impredicative type of propositions, from the calculus of constructions, with the inductive types and hierarchy of type universes of Martin-Löf's constructive type theory. Intuitively there is an easy way to determine an upper bound on the proof-theoretic strength. This is to use the 'obvious' types-as-sets interpretation of these type theories in a strong enough classical axiomatic set theory. The elementary forms of type of Martin-Löf's type theory have their familiar set-theoretic interpretation, the impredicative type of propositions can be interpreted as a two-element set and the hierarchy of type universes can be interpreted using a corresponding hierarchy of strongly inaccessible cardinal numbers. The assumption of the existence of these cardinal numbers goes beyond the proof-theoretic strength of ZFC. But Martin-Löf's type theory, even with its W types and its hierarchy of universes, is not fully impredicative and has proof-theoretic strength way below that of second-order arithmetic. So it is not clear that the strongly inaccessible cardinals used in our upper bound are really needed. Of course the impredicative type of propositions does give a fully impredicative type theory, which certainly pushes up the proof-theoretic strength to a set theory, Z^- , whose strength is well above that of second-order arithmetic. The hierarchy of type universes will clearly lead to some further strengthening. But is it necessary to go beyond ZFC to get an upper bound? Surprisingly perhaps, the types-as-sets interpretation has hardly been studied systematically. So it is the main aim of this paper to start such a systematic study. In Section 2 we first present some of the details of the TS interpretation of a type theory MLW^{ext} that is a reformulation of Martin-Löf's extensional type theory with W types but no type universes. This interpretation is carried out in the standard axiomatic set theory ZFC and so gives a proof-theoretic reduction of MLW^{ext} to ZFC. Of course this result is much too crude and we go on to describe two approaches to getting a better result. The first approach is to make the type theory classical by adding the natural formulation of the law of excluded middle. It turns out that to carry through the interpretation we need to strengthen the set theory by adding a global form of the axiom of choice and we get a proof-theoretic reduction of $MLW^{ext} + EM$ to ZFGC. Fortunately, it is known that the strengthened set theory is not proof-theoretically stronger, so that we do get a reduction of $MLW^{ext} + EM$ to ZFC. Section 2 ends with the second approach, which is to replace the classical set theory by a constructive set theory, CZF^+ , that is based on intuitionistic logic rather than classical logic. So we get a reduction of MLW^{ext} to CZF^+ . In section 3 we extend the results of section 2 by adding first a type universe reflecting the forms of type of MLW^{ext} and then an infinite cumulative hierarchy of such type universes. To extend the TS interpretation to the resulting type theories we use, in classical set theory, strongly inaccessible cardinal numbers for the type theories with EM, and in constructive set theory, inaccessible sets as introduced by *M. Rathjen, E. R. Griffor and E. Palmgren* [Ann. Pure Appl. Log. 94, No. 1-3, 181-200 (1998; Zbl 0926.03074)]. Finally in section 3, we formulate type theories having rules for the impredicative type of propositions of the calculus of constructions and formulate corresponding axioms of constructive set theory and again describe how each of these type theories has a TS interpretation into a corresponding set theory. In section 4 we briefly describe how the sets-as-trees interpretation of CZF into the type theory MLWU extends to the other set theories, giving reductions to the corresponding type theories with an extra type universe. Fortunately, each type theory with an infinite hierarchy of type universes is proof-theoretically as strong as the type theory with a type universe added on top, so that we end up with results stating that to each of the type theories we consider that have an infinite hierarchy of type universes there is a corresponding set theory of the same proof-theoretic strength. In particular, the type theory $MLWPU_{<\omega}$, that is our approximation to the type theories implemented in Lego and Coq, has the same proof-theoretic strength as the set theory $CZF^+pu_{<\omega}$. This last result does not solve the original problem motivating our work as the set theory is unfamiliar. Nevertheless, I think that it does give a new handle on the problem. The new set theory is an interesting one and I plan to present some results about it on a future occasion.

Keywords: proof-theoretic strength; proof development systems Lego and Coq; constructive type theory; types-as-sets interpretation; constructive set theory; type universe; classical set theory; inaccessible cardinal; inaccessible sets; calculus of constructions; sets-as-trees interpretation